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THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF THE NONLINEAR DYNAMIC RESPONSE OF CONTINUOUS SYSTEMS

Thein Wah

FINAL REPORT

January 31, 1965

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Department of Structural Research

SOUTHWEST RESEARCH INSTITUTE

SAN ANTONIO HOUSTON

SOUTHWEST RESEARCH INSTITUTE
8500 Culebra Road, San Antonio, Texas 78206

THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF THE NONLINEAR DYNAMIC RESPONSE OF CONTINUOUS SYSTEMS

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Robert C. DeHart

Robert C. DeHart, Director
Department of Structural Research

ABSTRACT

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This report describes experimental and theoretical work on the nonlinear vibration of uniform beams for various support conditions. The experimental work is an account of an attempt to extend the Moiré method to dynamic problems by use of high speed photography. The theoretical work is mainly the development of a numerical procedure for investigating vibration of nonlinear beams for those cases where the space and time variables are not separable.

Author

ACKNOWLEDGEMENTS

The author wishes to record his deep appreciation of the careful and painstaking experimental work and subsequent reduction of data carried out by Clayton R. Franz, Research Assistant.

Grateful acknowledgments are due to Dr. J. E. Prince of the Aerospace Medical Center, Brooks Field, Texas, for allowing the use of the motion analyzer in his department.

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I. INTRODUCTION

The purpose of this program was to study, both theoretically and experimentally, the behavior of continuous elastic systems such as beams and plates when vibrating with amplitudes so large that the linear theory for these systems is no longer applicable, but still sufficiently small to ensure that the material behaved elastically at all times.

The differential equations of motion of such systems can be written without great difficulty. They are, of course, nonlinear partial differential equations. The techniques for solving such equations are, however, extremely meagre, and one is more or less thrown on his own resources to obtain a solution. It is no exaggeration to say that almost none of these equations admit of an exact theoretical solution. Two possible approaches exist for the solution of these differential equations. The first is an approximate analytical solution, and the second is a numerical step by step procedure. The writer has preferred the first approach and has been successful in devising some simple but approximate solutions which have already appeared in the literature[1,2,3]. However, he has found it almost impossible to improve on these solutions and to extend these methods to more complicated problems. He has become convinced that the only practical procedure at the present time is to use high speed digital computers together with suitable numerical procedures.

The basic difficulty lies in the reduction of the system to one described by an ordinary nonlinear differential equation. The simplest analytical procedure is to separate the variables, but this is generally impossible to do exactly except for the very simplest equations. When numerical integration procedures are employed, however, one essentially supposes that the motion of the system can be described by an infinite number of linear partial differential equations, each equation being valid for a suitably chosen small interval of time Δ. This procedure has been described in a previous report but will be included here below again for completeness.

The present investigators believe that a vast accumulation of theoretical solutions is of little value unless a serious attempt is made to verify them by experimental observation of real systems. Apart from the feeling of confidence this could engender, it is believed that the feedback from experimental work would help in advancing the theoretical concepts. Primary emphasis was therefore placed in the development of experimental techniques for studying nonlinear vibrating systems, and this report is therefore largely a description of these attempts.

The program was limited to studying the vibration of beams held so that during motion no displacement of the ends is permitted. This restraint induces an axial tension in the beam and results in a nonlinearity of the governing differential equation.

II. THEORETICAL WORK

The theoretical work was confined to developing a numerical procedure for solving nonlinear beam problems for those cases where variables are not separable. In particular the concept of "normal modes" was extended to such cases.

The writer has pointed out[1] that certain nonlinear continuous systems, characterized by the separability of the space and time variables, vibrate in normal modes in the sense defined by Rosenberg[4]. In particular, the writer has shown that a simply-supported beam rigidly held at its ends is capable of vibrating in normal modes.

The motion of beams with axial tension is governed by the differential equation

$$EI \frac{\partial^4 w}{\partial x^4} - N \frac{\partial^2 w}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

with

$$\frac{Na}{AE} = \frac{1}{2} \int_0^a \left(\frac{\partial w}{\partial x} \right)^2 dx$$

in which N is the axial force, w is the lateral deflection, ρ the mass density per unit length, a the length of the beam, A its cross-sectional area, E is Young's modulus, I is the moment of inertia, and t is time.

In the case of a simply supported beam, a sine function in the space coordinate effectively separates the variables in (1), as is well known, and normal modes emerge very simply. As far as the writer is aware, this separability is confined to the simply supported case.

It is certainly very surprising that normal modes cannot be readily defined for nonsimply supported beams. It is difficult to accept the conclusion that the concept breaks down completely when the boundary conditions are changed. Consequently, it becomes necessary to generalize this concept so as to take into account physical systems in which variables are not separable.

While the possibility of such theoretical extension exists, it is proposed to investigate,first, by an approximate numerical procedure whether such normal modes, conceived of intuitively, are stable.

A. Time Dependent Normal Modes

If one recalls the definition of normal modes in linear oscillations, it is evident that they can occur only under certain, quite restrictive, initial conditions. Thus, a beam will not vibrate in a normal mode unless it is started exactly in the appropriate mode, which, in turn, depends on its boundary conditions. A beam which is given an arbitrary starting shape will not vibrate in a normal mode. While the concept itself may be of considerable practical value, the point made here is that normal modes require restrictive conditions for their actual occurrence.

Returning now to the system governed by equations (1), the simply supported case (where the variables separate) is characterized by the fact that the mode shape itself is independent of time,being always a sine function. In the case of other boundary conditions, since variables do not separate, one must suppose that the normal modes are a function of time, the mode shape altering continuously as the beam vibrates. However, if normal modes exist, they must be stable; that is, the beam must return to a certain shape periodically. If it does not, then the normal modes are unstable.

B. Numerical Procedure

In order to investigate analytically whether such time dependent normal modes are stable, one must appeal to physical intuition for certain basic assumptions. We suppose first that the axial tension N in (1) does not vary very rapidly so that it may be assumed to remain constant for sufficiently small intervals of time. If N is constant in the first of equations (1), it is linear and normal modes exist, together with a denumerably infinite sequence of discrete frequencies for any set of suitable boundary conditions. At the end of the time interval, one may use the second of equations (1) to calculate a new membrane tension and a new set of normal modes and natural frequencies. Physical intuition suggests that the first normal mode (that is, the normal mode corresponding to the lowest frequency) of the first time interval will merge into the first normal of the second time interval and so on for all subsequent time intervals and other normal modes. This is a fundamental assumption of the investigation.

Since normal modes depend on the initial conditions, one must specify such conditions consistently. Assuming that the beam is initially displaced with zero initial velocity, the normal modes are completely defined by the tension N.

For a constant N, one may assume that

$$w = X(x) \exp(ipt) \tag{2}$$

where X is a function of x alone.

The initial condition is specified by the tension parameter

$$\mu = \frac{Na^2}{2EI} = \frac{Aa}{4I} f^{*2} \int_0^a (X')^2 dx \tag{3}$$

in which f*, the amplitude of the displacement, is unknown to begin with, A is the cross-sectional area and prime denotes differentiation with respect to x.

For purposes of computation, it is convenient to write the relation (3) in the form

$$\mu = \lambda \frac{N^*a^2}{2EI} = \frac{Ah^2}{4I} \cdot f^2 \phi \tag{4}$$

with

$$\begin{aligned} \lambda &= N/N^* \\ f &= f^*/h \\ \phi &= a \int_0^a (X')^2 dx \end{aligned} \tag{5}$$

where N* is the "buckling load" of the beam treated as a column and h is any convenient dimension of the beam cross section, such as its depth.

If we let

$$\begin{aligned} \alpha &= [\mu + (u^2 + \mu^2)^{1/2}]^{1/2} \\ \beta &= [(u^2 + \mu^2)^{1/2} - \mu]^{1/2} \end{aligned} \tag{6}$$

where

$$u^2 = \rho \frac{a^4 p^2}{EI}$$

then it may be shown by elementary procedures that for beams clamped at $(x = 0, a)$ and for beams clamped at $x = 0$ and simply supported at $x = a$, the normal modes $X(x)$ are given by

$$X(x) = \frac{\cosh \frac{\alpha x}{a} - \cos \frac{\beta x}{a}}{\cosh \alpha - \cos \beta} + \frac{\beta \sinh \frac{\alpha x}{a} - \alpha \sin \frac{\beta x}{a}}{\alpha \sin \beta - \beta \sinh \alpha} \quad . \tag{7}$$

The frequency equations for the two cases are, however, different.

For the clamped-clamped beam, the equation is

$$u\,(1 - \cosh \alpha \cos \beta) + \mu \sinh \alpha \sin \beta = 0 \tag{8}$$

and for the clamped simply-supported beam, the frequency equation is

$$\alpha \sin \beta \cosh \alpha - \beta \cos \beta \sinh \alpha = 0 \tag{9}$$

For a given μ, α, β and u must satisfy equations (6) and (8) or (9). The values of α, β and u may thus always be obtained by trial and error. With α and β known, the mode shape is completely defined by (7). It is to be noted, however, there are an infinity of the set (α, β, u) for a given μ. We are, for the present, interested only in the set corresponding to the lowest value of u.

Consider any time interval extending from $r\Delta$ to $(r+1)\Delta$, where Δ is a suitably small quantity. At time $r\Delta$, the displacement and velocity are given by

$$\frac{w}{h} = X_{r-1}\,[f_{r-1} \cos (p_{r-1}\Delta) + g_{r-1} \sin (p_{r-1}\Delta)] \tag{10}$$

$$v = \frac{\dot{w}}{h} = X_{r-1}\, p_{r-1}\,[-f_{r-1} \sin (p_{r-1}\Delta) + g_{r-1} \cos (p_{r-1}\Delta)] \tag{11}$$

The tension parameter is:

$$\mu_r = \frac{Ah^2}{4I} [f_{r-1} \cos (p_{r-1} \Delta) + g_{r-1} \sin (p_{r-1} \Delta)]^2 \phi_{r-1} \tag{12}$$

Knowing μ_r, one may calculate α_r, β_r, p_r and X_r. The displacement, velocity and membrane tension for $(r\Delta) \leqslant t \leqslant (r+1)\Delta$ are then given by

$$\frac{w}{h} = X_r(f_r \cos p_r t + g_r \sin p_r t) \tag{13}$$

$$v = \frac{\dot{w}}{h} = X_r p_r (-f_r \sin p_r t + g_r \cos p_r t) \tag{14}$$

$$\mu_r = \frac{Ah^2}{4I} f_r^2 \phi_r \tag{15}$$

However, the displacement, velocity and membrane force at t = 0 as given by (13), (14), and (15) must be the same as those given by (10), (11) and (12) and so

$$X_{r-1} [f_{r-1} \cos (p_{r-1} \Delta) + g_{r-1} \sin (p_{r-1} \Delta)] = X_r f_r \tag{16}$$

$$X_{r-1} p_{r-1} [-f_{r-1} \sin (p_{r-1} \Delta) + g_{r-1} \cos (p_{r-1} \Delta)] = X_r p_r g_r \tag{17}$$

$$[f_{r-1} \cos (p_{r-1} \Delta) + g_{r-1} \sin (p_{r-1} \Delta)]^2 \phi_{r-1} = f_r^2 \phi_r \tag{18}$$

From (18), we get

$$f_r = [f_{r-1} \cos (p_{r-1} \Delta) + g_{r-1} \sin p_{r-1} \Delta] \left(\frac{\phi_{r-1}}{\phi_r}\right)^{1/2} \tag{19}$$

From (15) and (19)

$$\frac{X_{r-1}}{X_r} = \left(\frac{\phi_{r-1}}{\phi_r}\right)^{1/2}$$

Substitution in (16) yields

$$g_r = \frac{p_{r-1}}{p_r} [-f_{r-1} \sin (p_{r-1} \Delta) + g_{r-1} \cos (p_{r-1} \Delta)] \left(\frac{\phi_{r-1}}{\phi_r}\right)^{1/2} \tag{20}$$

The relations (19) and (20) thus give f_r and g_r in terms of all the known quantities and enable one to proceed to the next step. It is important in this computation that the time steps be taken as small as practical,as otherwise the results will indicate an unstable motion even when it is essentially stable.

Although when the beam is simply supported, an exact solution in terms of elliptic functions can be obtained, it is of interest to apply the numerical method to this case also to afford a comparison with the solutions for other boundary conditions.

Figure 1 shows a plot of the center deflection ratio (w/h) against the velocity/frequency ratio (v/h)/p, which may be called a modified phase plane. If the motion considered is stable and periodic, a closed curve should result.

The parameters chosen in the computations were those of the steel specimens to be used in the experimental work and are as follows:

length = 30 in. depth = 0.25 in. width = 0.30 in.

The resulting nondimensional parameters are

$$A_1 = (EI/a^4\rho)^{1/2} = 19.456$$

$$A_2 = (Ah^2/4I) = 3$$

The parameter λ was taken as 2.

In Figure 1 it may be noted that all the curves are essentially closed, although not exactly so. Extended computations, with various values of the time interval show, however, that the gap is a function of the time interval and decreases as the interval decreases. In the computations the quantity $(p\triangle)$ was taken as .005,and thus over 1000 intervals were required to

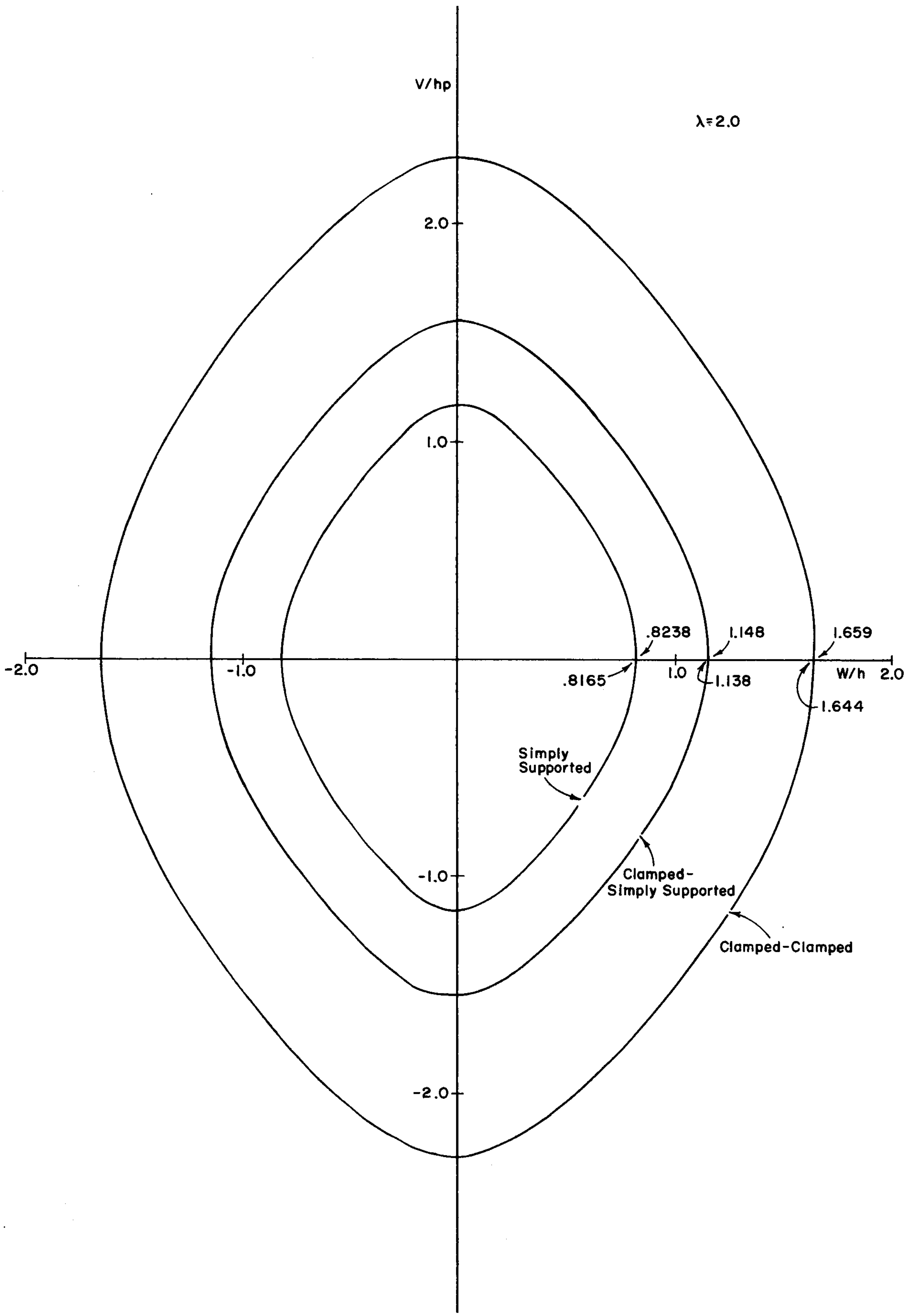

FIGURE 1.

PLOT OF DISPLACEMENT RATIO AGAINST VELOCITY RATIO/FREQUENCY complete 2π radians. A further decrease in the time interval seemed unwarranted, especially as it has been established that any slight gap at closure was an approximation error.

We conclude therefore that the motion is stable and periodic for all boundary conditions, and that it is meaningful to speak of a time dependent normal mode.

C. <u>Period of "Fundamental" Mode</u>

The nonlinear period T^* of vibration for the "fundamental" mode may be written in the following form

$$T^* = 4K/p \tag{21}$$

$$p = \alpha^2 A_1 \left(1 + \nu A_2 \frac{f_o^2}{h^2}\right)^{1/2} \tag{22}$$

$$m = \frac{1}{2}\, \nu A_2 \frac{f_o^2}{h^2} \Big/ \left(1 + \nu A_2 \frac{f_o^2}{h^2}\right) \tag{23}$$

in which K is the complete elliptic integral of the first kind, p is the frequency, f_o is the initial displacement and m is the parameter of the Jacobian elliptic functions. The parameters α and ν have the following values for the various boundary conditions:

(a) Simply supported

$\alpha =$ [illegible], $\nu = 1$

(b) Clamped-clamped

$\alpha = 4.73$, $\nu = .3077$

(c) Clamped-simply supported

$\alpha = 3.927$, $\nu = 1.6697$

Formulas (21), (22) and (23) are theoretically exact only for the simply supported case. For the other two cases they are approximate, having been arrived at by applying a Galerkin approximation to the differential equation[3].

Table 1 gives a comparison of the fundamental period as arrived at by formula (21) and by the numerical procedure outlined above.
It will be noted that the agreement is close for the simply supported case only. For the other boundary conditions, formula (21) underestimates the period as might be expected.

TABLE 1

FUNDAMENTAL PERIOD T* (IN SECS)

	SS	CC	CS
Formula (21)	.02088	.00860	.00878
Num. Proc.	.02080	.00934	.01348

SS = Simply supported

CC = clamped-clamped

CS = clamped-simply supported

III. AIMS OF EXPERIMENTAL WORK

The experimental work had two main objectives in view. Theoretical work has indicated[1] the existence of normal modes of vibration in nonlinear continuous systems, similar in many respects to that observed in linear systems. In particular, for simply supported beams such modes can be described with relative simplicity; but for other conditions of support, the nonlinear normal modes are difficult to describe analytically. A general experimental technique was required which could enable the observation of the shape assumed by a vibrating beam or plate continuously. A technique with considerable appeal because of its inherent simplicity was the so-called Moiré method[5,6], which has been applied with some success to study plates under static loads. It was felt that combined with high speed photography this method should provide a simple way of observing the behavior of vibrating beams and plates.

A second objective of the experimental work was considerably simpler; namely, to see whether the theoretically predicted nonlinear frequency of vibration could be verified experimentally. All these systems, to a first approximation, describe a motion which can be represented by elliptic sines and cosines[1,2,3]. However, unlike linear systems, the frequency of vibration is a function of the initial displacement.

For this second objective the Moiré method could be used but is not necessary, at any rate for beams. However, high speed photography was essential because of the difficulty of supplying an elliptic-sine type excitation to the system; and therefore the free vibrations tended to be damped out rapidly. The technique was to start the structure in a (theoretical) normal mode and to photograph the vibrating beam edgewise and then analyze the motion.

Both these methods were tried and are described in the following sections. All the experiments thus far carried out were on beams, since they constitute the simplest continuous system.

IV. MOIRÉ METHOD

The method of Moiré fringes has become useful, as previously noted, as an experimental tool in the analysis of structural components under static loads. However, this method has apparently been untried for the analysis of dynamic problems.

A photograph showing the general arrangement for the tests is shown in Figure 2.

The beam is positioned in a base plate (Figure 3) fabricated from a 3-1/2-inch thick steel plate with a 15-inch by 30-inch window. This plate, isolated from the floor and adjacent structures with one-inch neoprene padding to reduce random vibrations in the test specimen, is anchored to a concrete floor with two 1-3/4-inch bolts. Steel plates, one inch thick, are used to clamp the test specimen.

The grid (Figure 4) was made from a durable type cardboard lined with alternate black and white lines 0.1 inch in width.

The cardboard is mounted in a housing constructed on one-inch aluminum plate (Figures 2 and 4). Channels in this housing allow not only for placement of the grid parallel to the undeflected specimen but also afford placement in circles with radii of 2-1/2 and 3-1/2 times the length of the most distant part of the grid to the model surface. Such an arrangement permits experimentation with various patterns so as to optimize clarity and accuracy.

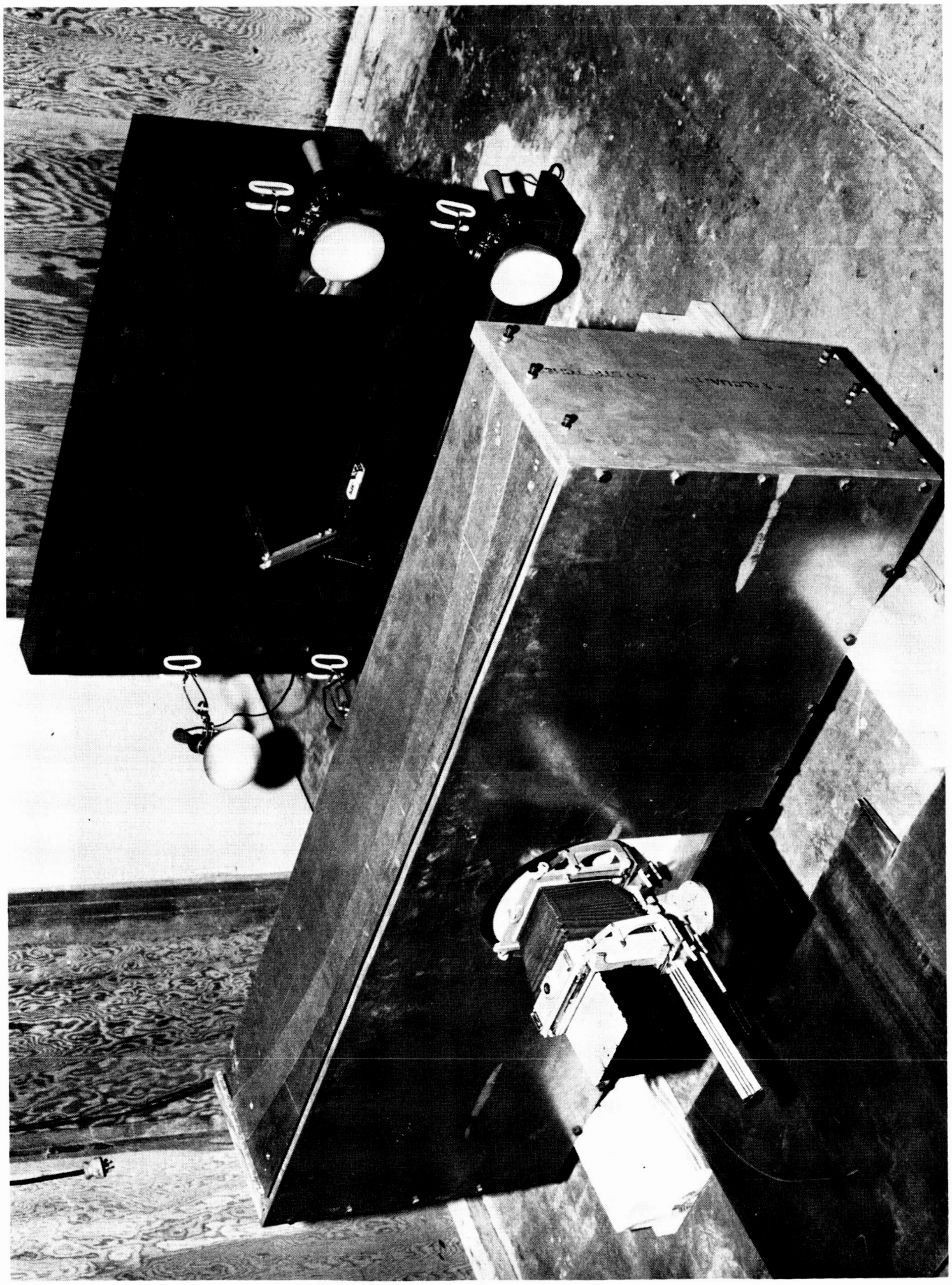

FIGURE 2. TEST ARRANGEMENT

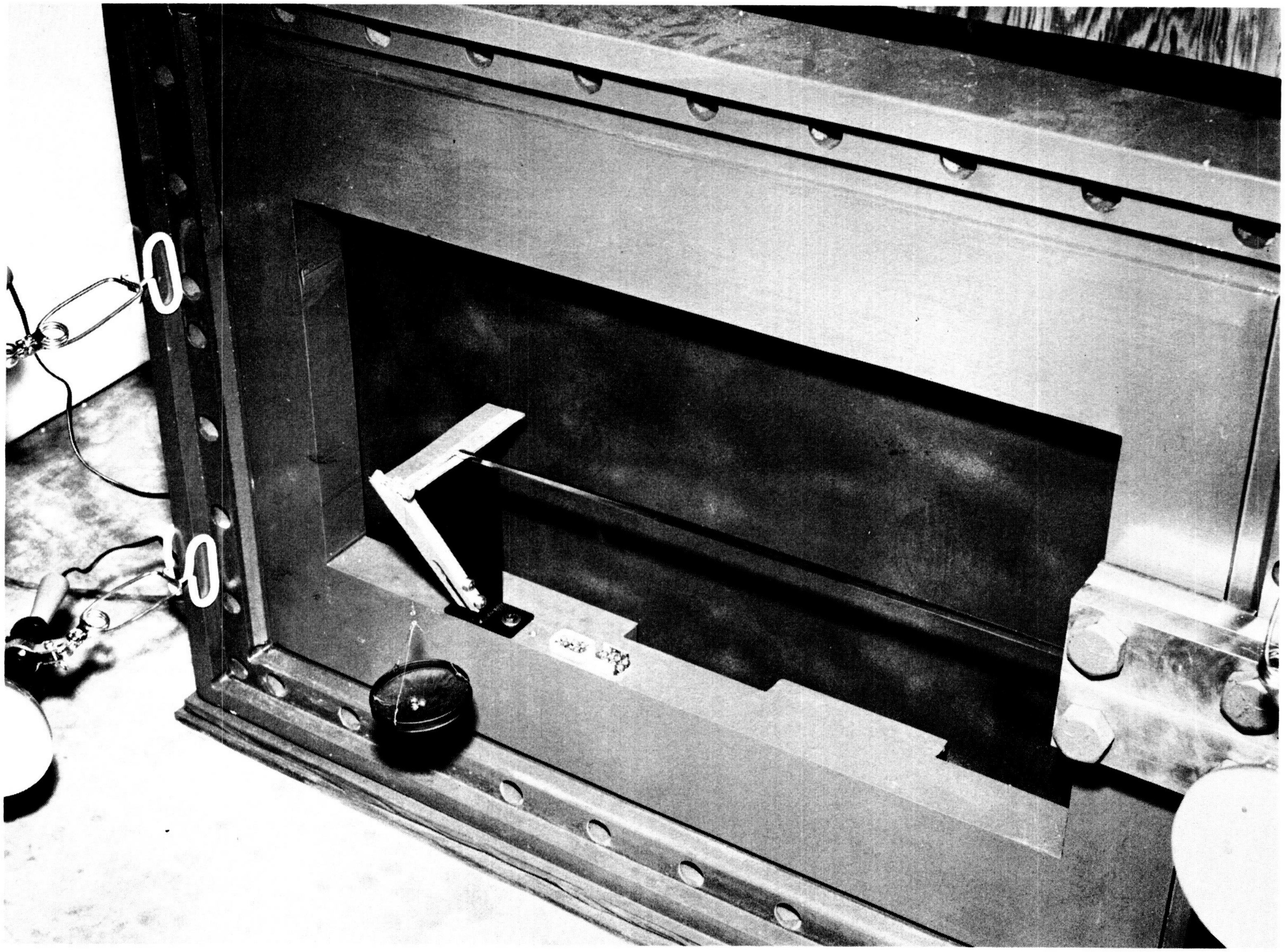

FIGURE 3. TEST SPECIMEN AND LOADING ARRANGEMENT

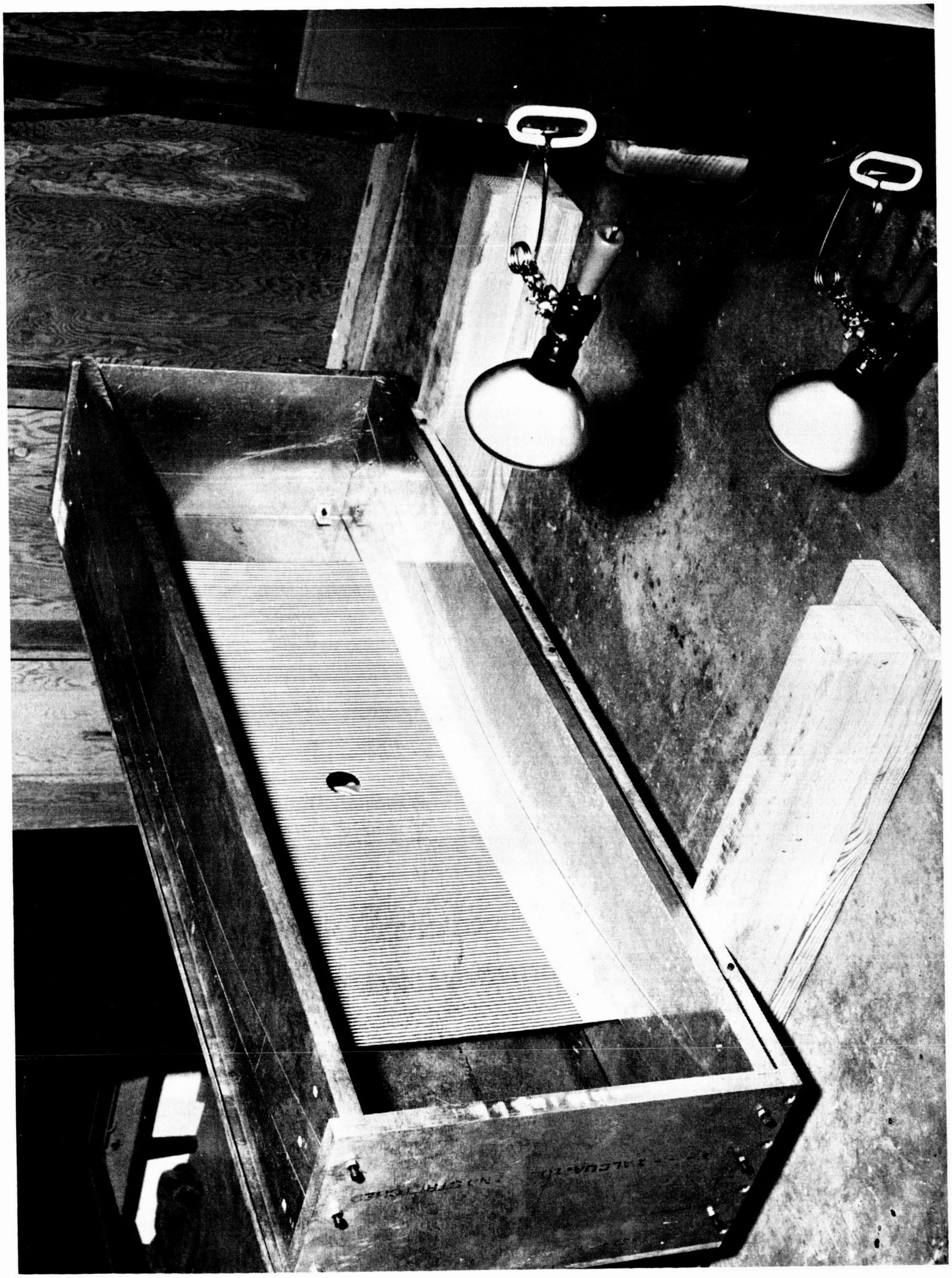

FIGURE 4. GRID AND HOUSING

The camera (Figure 2) is mounted behind the center of the grid and views the specimen and the reflected grid from the most distant part of the grid. Lighting is achieved by appropriate placement of flood lamps around the edge of the base plate.

The static experiments conducted for calibration purposes consisted of applying a known load to the end of a cantilever and comparing the actual deflection with that computed from the Moiré patterns. The accuracy of the results varied with the curvature of the grid used. Figure 5 (with the label "1") shows the reflected grid when the specimen is undeflected. Figure 6 (with the label "4") shows the fringe pattern with grid plane flat and the specimen loaded, and Figure 7 (with the label "5") shows the fringe pattern with the grid plane having a radius of curvature of 3.5d, where d is the distance of the grid from the model. The flat grid gave relatively poor results; whereas, the curved grid gave an excellent prediction of the maximum deflection (.498 inch as against a measured deflection of 0.5 inch).

It should be noted that for the static tests, a cantilever specimen was used (Figures 2 and 4); but, for the dynamic experiments, the test fixture was modified to accommodate simply supported and clamped beams.

Two beams of different configurations were designed--one for both ends clamped, the other for both ends pinned (Figure 8). Each beam is approximately 30 inches long, with a 0.250 inch or 0.300 inch cross section, fabricated from high strength steel plate, and having a .25-inch reflective surface machined to a No. 2 finish.

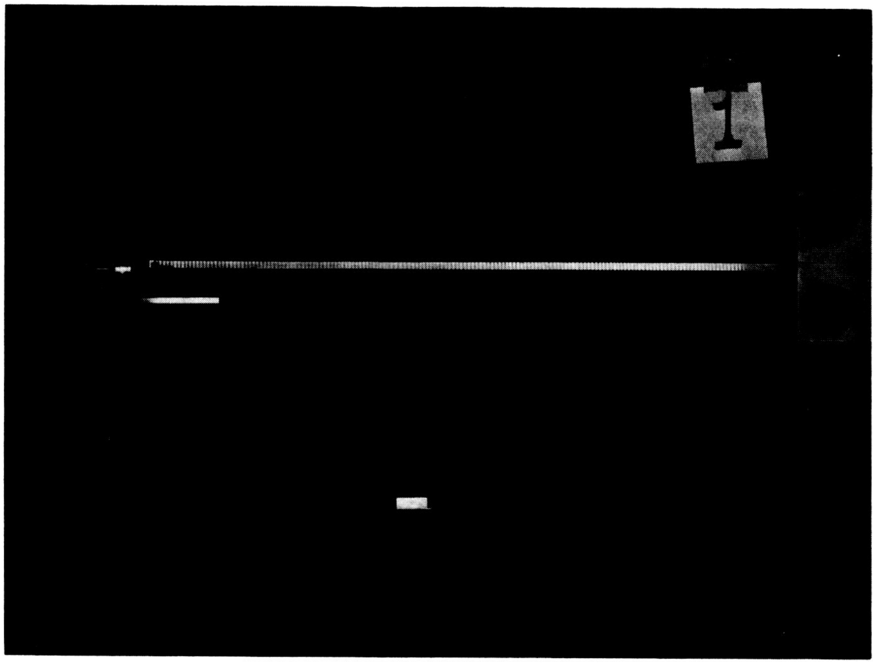

FIGURE 5. CANTILEVER IN UNDEFLECTED POSITION

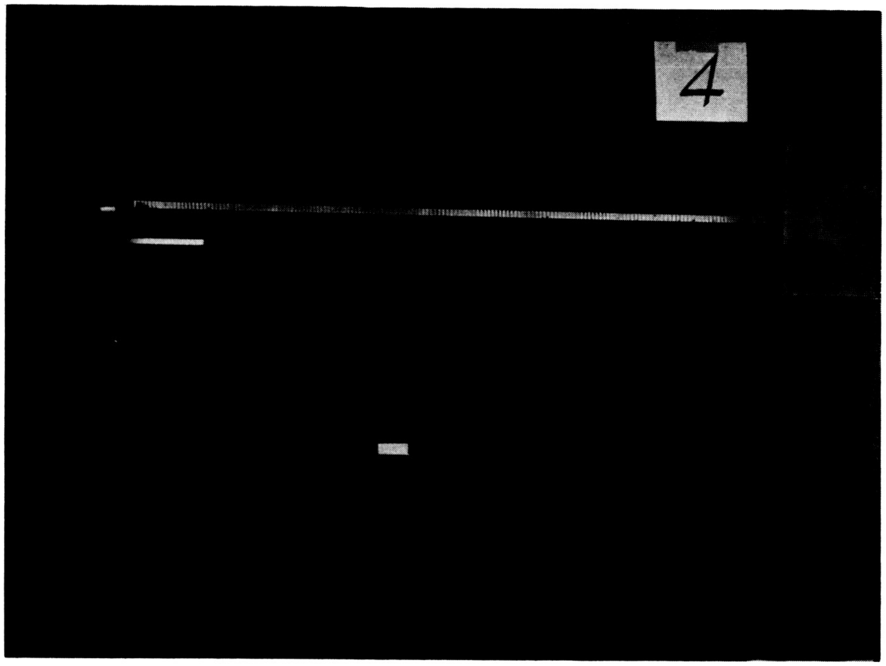

FIGURE 6. CANTILEVER DEFLECTED ONE-HALF INCH AT TIP, MOIRÉ PATTERNS WITH GRID FLAT

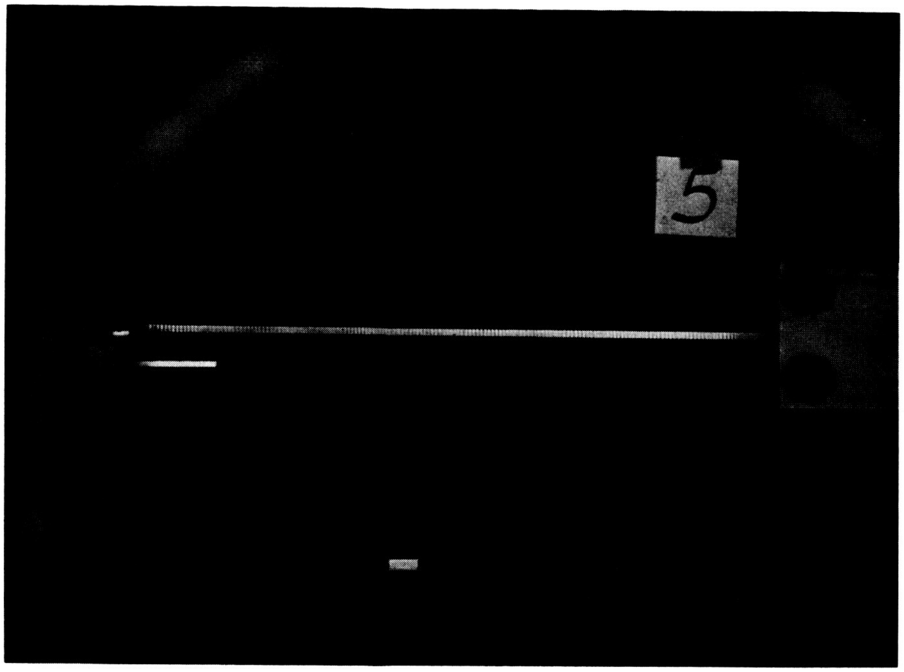

FIGURE 7. CANTILEVER DEFLECTED ONE-HALF INCH AT TIP, MOIRÉ PATTERNS WITH GRID RADIUS OF CURVATURE 3.5d (140 Inches)

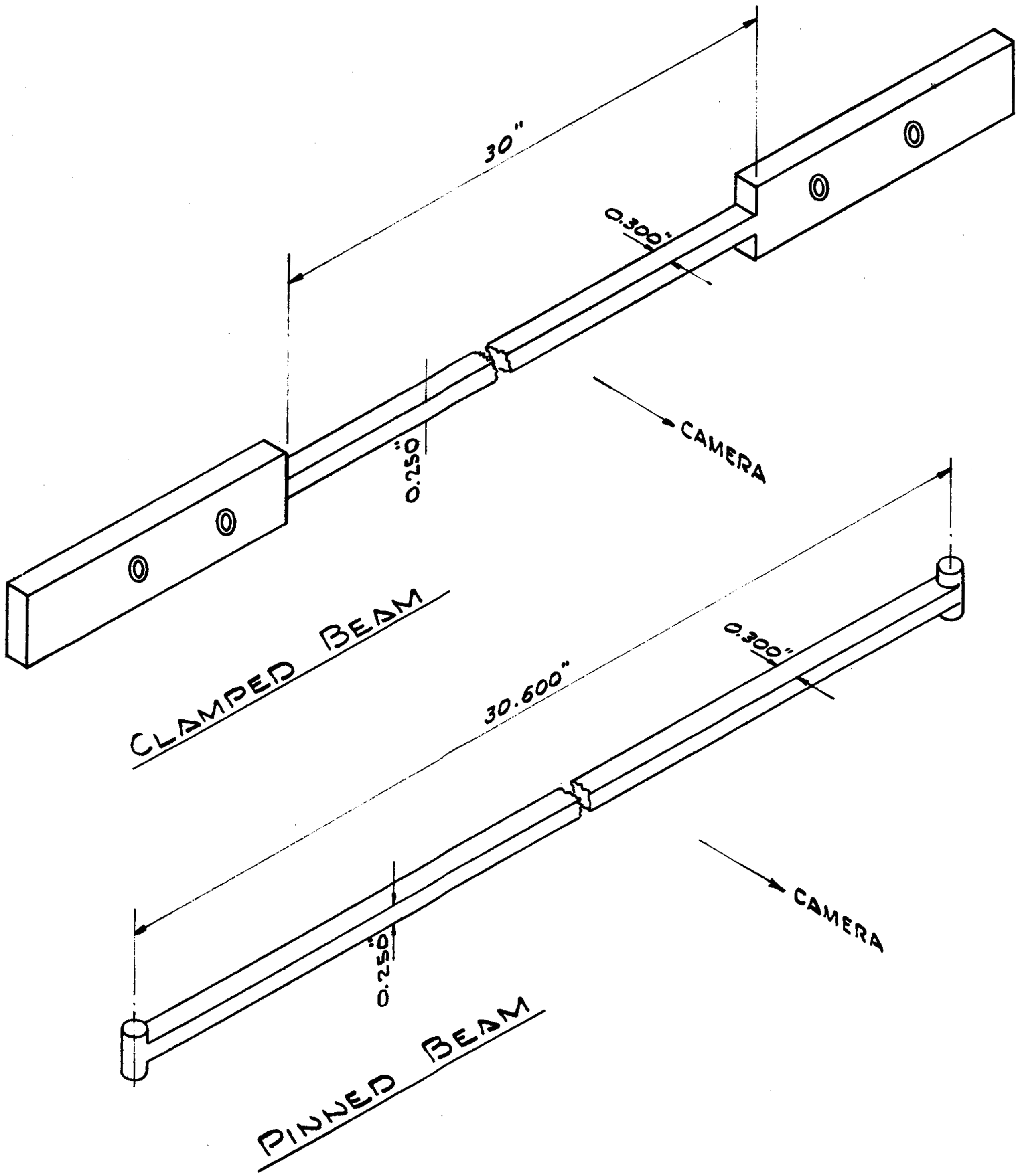

FIGURE 8. SKETCH OF BEAMS USED IN EXPERIMENTS

An extended effort was made to develop this technique into a practical experimental procedure for determining frequencies and mode shapes of nonlinear continuous vibrating systems. While experimenting with this method, several difficulties were encountered. Some of these were overcome, but others required extensive modification of equipment or additional purchases which proved prohibitive due to allotted time and funds.

A Fastex, 16 millimeter, high speed motion camera, capable of taking 8000 pictures per second, was used together with Wollensak "Goose" Control Unit. The "Goose" control unit provides a convenient means for synchronizing, in proper time relationships, the Fastex camera operation and the event being photographed. It also provides a means for safely increasing the voltage over that normally applied to the camera so that increased camera speeds may be obtained.

Tests revealed that the complexity of high speed photography in conjunction with the Moiré method made this a very difficult experiment. Resolution of the patterns recorded on some of the film, though visible, was very poor. This can be attributed to several factors.

High speed photography requires optimum illumination of subjects being photographed. Because of the technique employed by the Moiré method, only reflected light could be used in these experiments. When proper camera speed was used to record the action of the vibrating beam, the resulting motion film was dark with no pattern definition on the image of the model. A decreased camera speed revealed faint indication of the fringes, but the speed of motion of the fringes was excessive.

Normally, high speed film must be used when photographing motion at speeds found in a vibrating beam. Such films have a characteristically grainy texture. This tends to distort the resolution of the reflected grid lines, especially on as small an area as that afforded by the .250-inch wide beam.

Since the method of obtaining Moiré fringes necessitates rewinding of the film so that a double exposure of the beam (one before and one after it is set in motion) can be made, a slight change in the position of the film on its spool resulted in a relative displacement of the images of the beam in the two exposures. Subsequent tests with a still camera revealed that the double image of the beam could be eliminated by substituting a wider surface mirror in place of the beam on the initial exposure, thereby simulating the grid lines on the beam in its neutral position. The second exposure was then made of the statically deflected beam so that only one picture of the beam itself was taken.

When applying this technique to the dynamic situation, however, the camera failed to yield a film with resolution of the patterns such that a quantitative analysis could be made. Indications are that in addition to the difficulties already mentioned, use of two different reflective surfaces requires both to have the same reflective qualities. Otherwise, one overshadows the other and definition of the fringes is lost. Efforts to blend the two images on the same film by different lens settings were unsuccessful.

In selecting the material from which the model beam was to be fashioned, consideration had to be given to obtaining one which had or could be polished to a mirror-like finish. At the same time, the material had to be fabricated into a particular shape and further must possess sufficient yield strength so that its elastic behavior for large amplitudes was insured. High strength steel with a yield point of 80,000 psi was selected with one side lapped and polished to a No. 2 finish. This proved satisfactory when preliminary static tests were performed to check out apparatus. However, the dynamic tests revealed that a more reflective surface on the beam would be necessary for sufficient definition of the fringes.

In brief, the attempt to use the Moiré fringe method for vibrating beam problems has turned out thus far to be unsuccessful. This does not mean that the difficulties are insuperable. A necessary condition for successful application of the method seems to be the availability of high speed film with a very fine grain. It appears also that the technique would be more successfully applied to plates than to beams in which the narrowness of the reflecting surface inevitably renders the delineation of the fringe patterns very difficult. However, this could possibly be overcome with a fine grained film.

V. NATURAL FREQUENCIES OF NONLINEAR BEAMS

The Moiré method is not necessary when one is interested only in the vibration of beams. One could theoretically obtain much useful information by photographing the vibrating profile of the beam.

To achieve this,the position of the camera was changed such that the beam could be photographed from above to record the actual deflections during vibration (Figure 9). The beam was painted black with white lines indicating pertinent points along its length to be measured for deflection during the vibration cycles. Surrounding fixtures and background were painted white to accentuate the contrast with the image of the beam on the film. Five photo flood lamps were focused on the beam to supply necessary light to photograph the beam during motion using a camera setting of 5000 frames per second. The Fastex 16-MM camera loaded with Kodak Tri-X high speed reversal film was again used to conduct these experiments.

Positioning of the beam in the desired mode shape was achieved by wedging a lightweight aluminum load plate with 7 adjustable screws, placed to correspond with the white lined points on the beam, between a backup assembly and the beam (Figure 10). The backup assembly, which could be adjusted horizontally to deflect the model the proper distance, consisted of a 1/4"×2"×2"× 30 inch long angle and a 2-1/2"×1-1/2" × 20 inch long spacer plate (Figure 11). The screws, mounted in the load plate and with round heads to reduce friction, were adjusted so that their ends formed seven points of the predetermined mode shape of the beam.

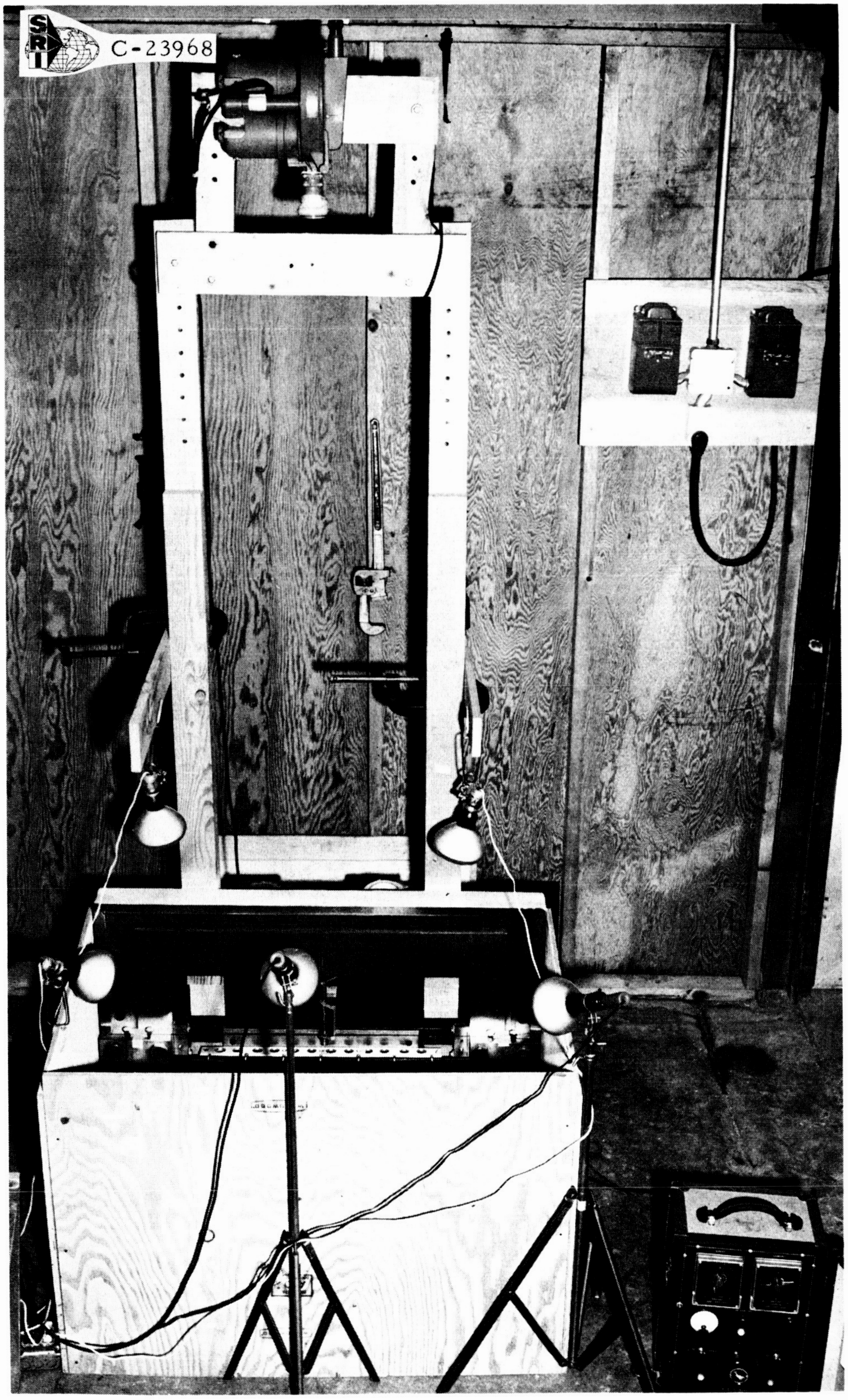

FIGURE 9. COMPLETE ASSEMBLY OF APPARATUS USED IN EXPERIMENTAL WORK TO DETERMINE NONLINEAR PERIOD

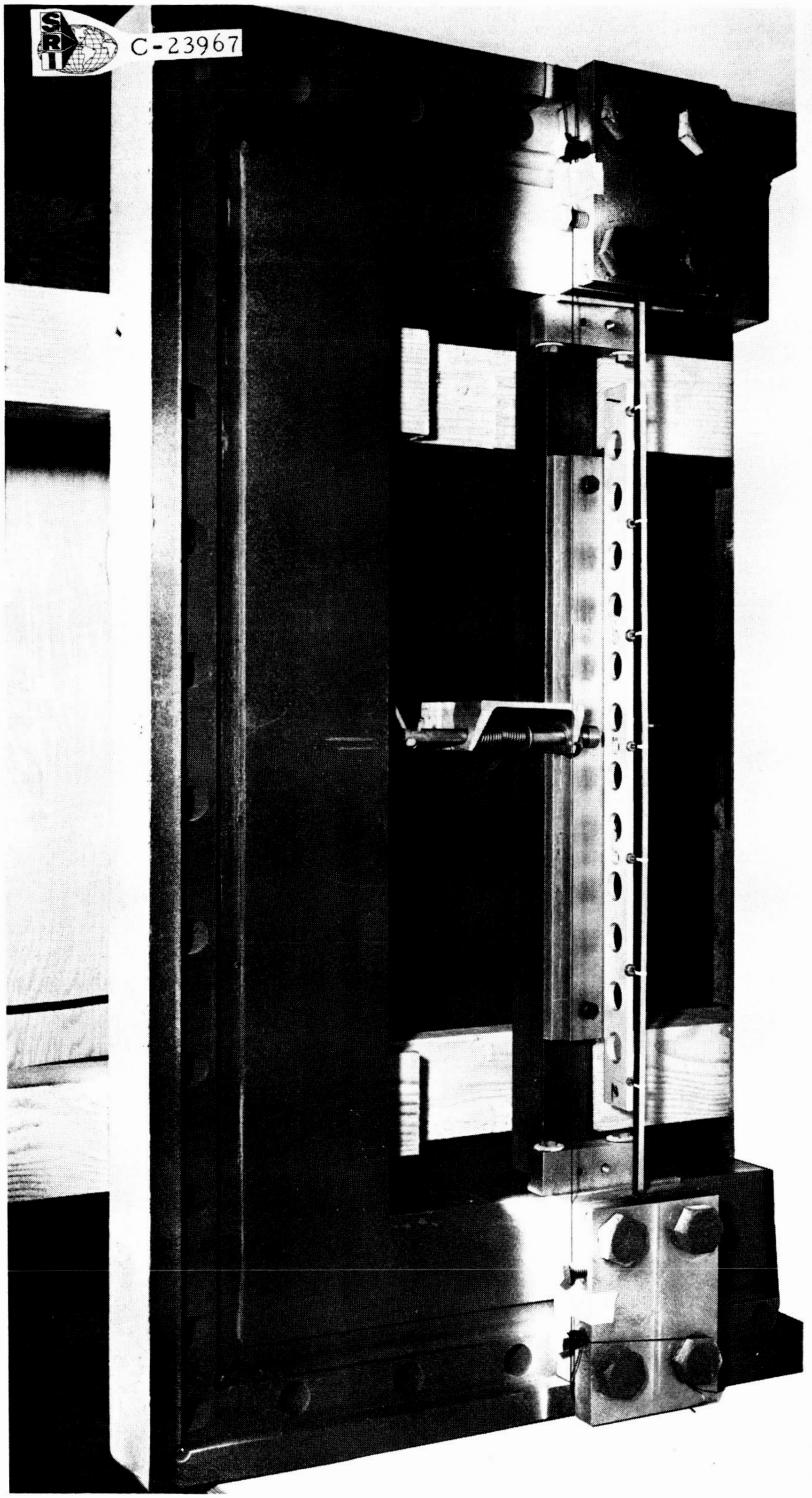

FIGURE 10. CLAMPED-CLAMPED BEAM INITIALLY DISPLACED AND READY FOR RELEASE

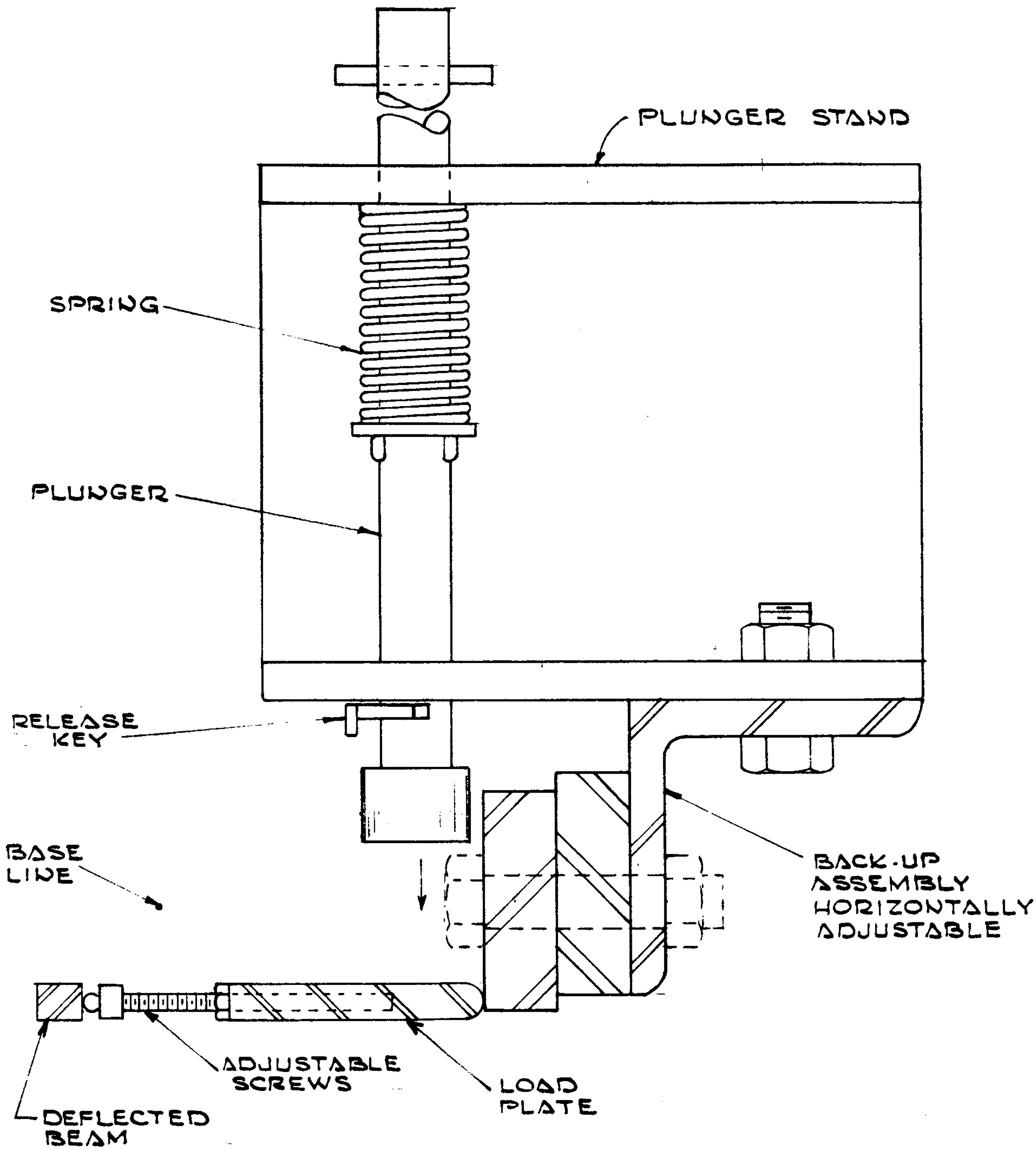

FIGURE 11. SKETCH OF SPRING LOADED PLUNGER ASSEMBLY COCKED AND READY TO STRIKE LOAD PLATE

The model was set in vibratory motion by sudden withdrawal of the load plate. This was achieved by a spring-loaded plunger (Figure 11), which when released would strike the edge of the load plate nearest the backup assembly, thereby knocking it away and allowing the model to vibrate freely. Camera start and plunger release were synchronized to allow acceleration of the film from 0 to 5000 frames per second.

The processed film was analyzed using a Traid Motion Analyzer, which permitted frame by frame inspection. Deflection of the beam at various time intervals in the first 2 or 3 cycles was measured at each of the seven points along the beam using a base line which simulated one edge of the beam in its neutral position. The period of the cycle was determined by multiplying the number of frames involved in one complete cycle by the time interval between frames. These data were then reduced and compared with the theoretical formulae.

The inherent difficulties of this method of setting the beam in vibration are obvious. The first difficulty lay in closely approximating the theoretical fundamental mode shape. While this is a simple half sine wave in the case of the simply supported beam, it assumes quite a complex mathematical form in the clamped case being dependent on the membrane tension. Adjusting the beam shape to conform to the theoretical form at seven points is still a crude approximation only, and inclusion in it of higher modes is unavoidable. This, of course, could greatly influence both the period of vibration and the mode of the vibration. Secondly, this method is restricted by the rapid decrease in the amplitude of the motion, so that one is limited to studying the first two or three cycles of the motion in order to obtain information on large amplitude vibrations.

Altogether, eleven (11) tests were conducted for the simply supported and for the clamped-clamped case. Only the amplitude of the initial displacement was varied in the different cases. Two types of information were sought, namely, the mode shape and the period of vibration. An essential assumption, of course, is that if the initial starting shape is not a true mode, the subsequent vibration will not persist in that mode for simply supported beams or return to that mode for clamped beams. Obviously, in the case of clamped beams, the agreement with theory cannot be expected to be particularly good because of the change of amplitude from cycle to cycle.

It has not been possible to reproduce information pertaining to all of the tests. The results of a few of the tests are presented in Figures 12 to 18 together with a comparison with theoretical data.

Figures 12 and 13 show the motion of the center-point of the beam for the clamped-clamped case and the simply supported case. It will be noted that in both cases the experimental beam had a smaller frequency of motion. The damping out of the motion is observable in the reduction of the amplitude of the displacement.

Figures 14 to 16 show the mode shapes assumed by the beam at the times noted. The mode shapes are "normalized", that is, all ordinates are divided by the maximum ordinate of that curve.

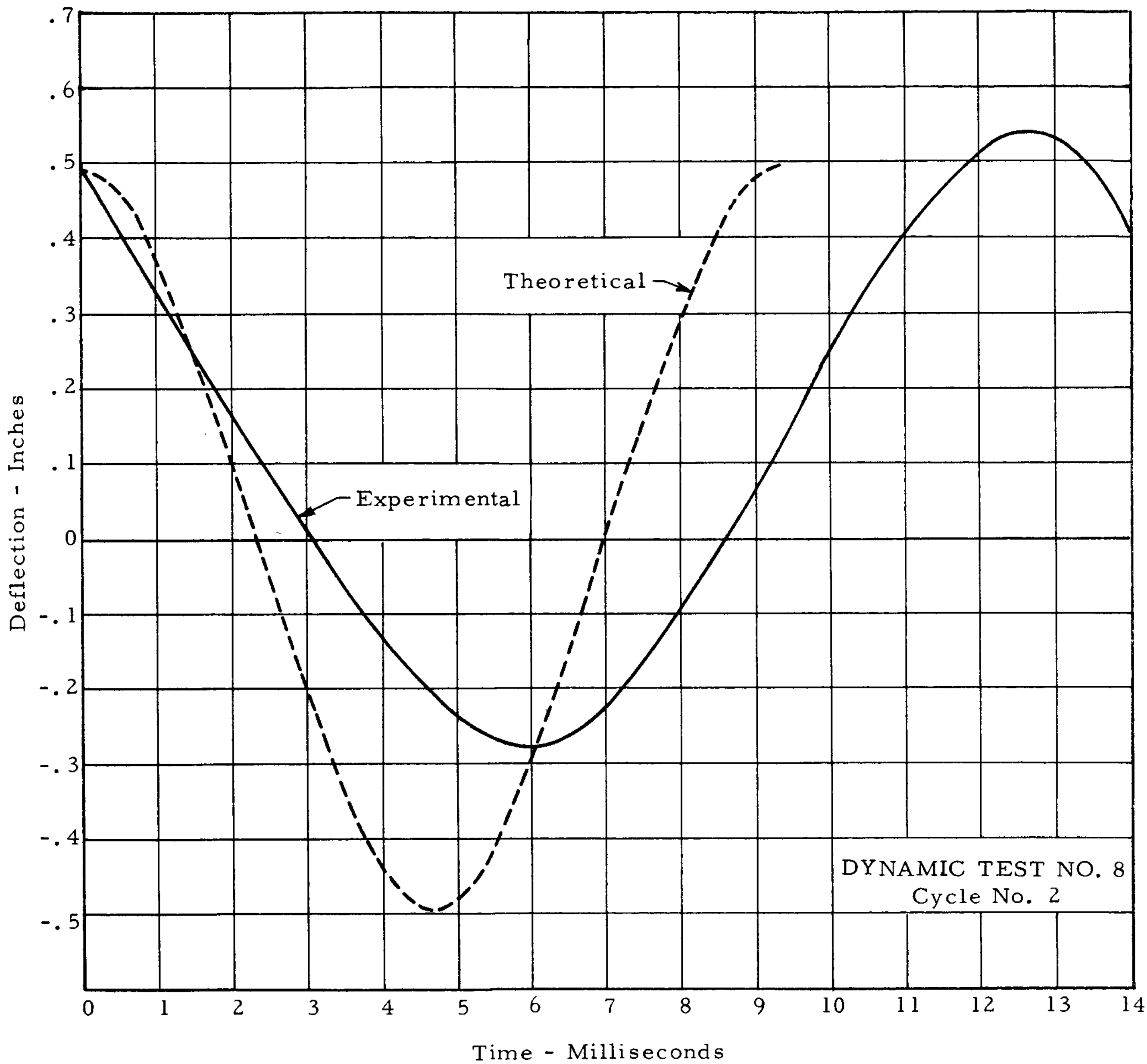

FIGURE 12. MOTION OF MIDSECTION-CLAMPED BEAM

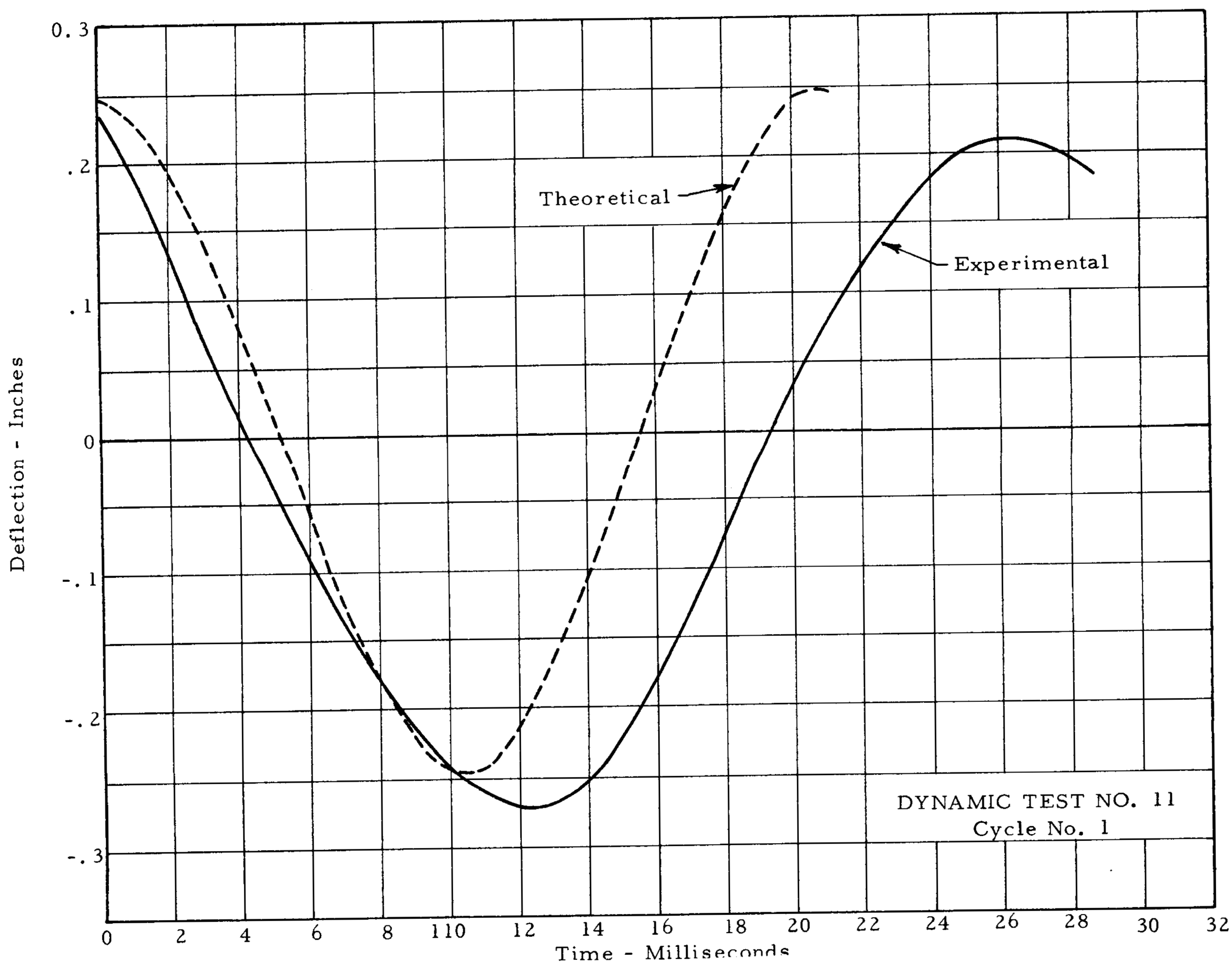

FIGURE 13. MOTION OF MIDSECTION-SIMPLE BEAM

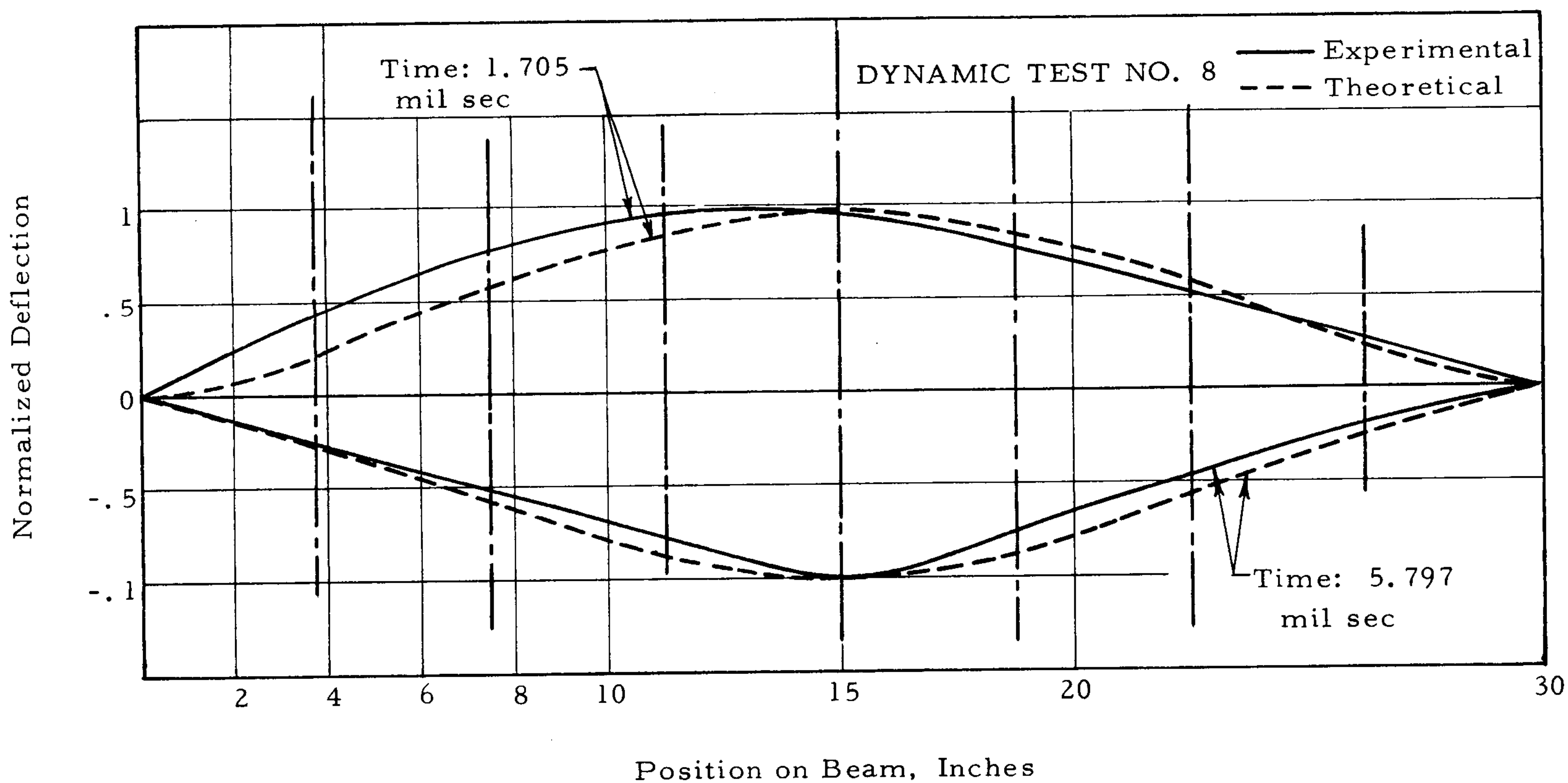

FIGURE 14. SHAPE OF VIBRATING BEAM-CLAMPED BEAM

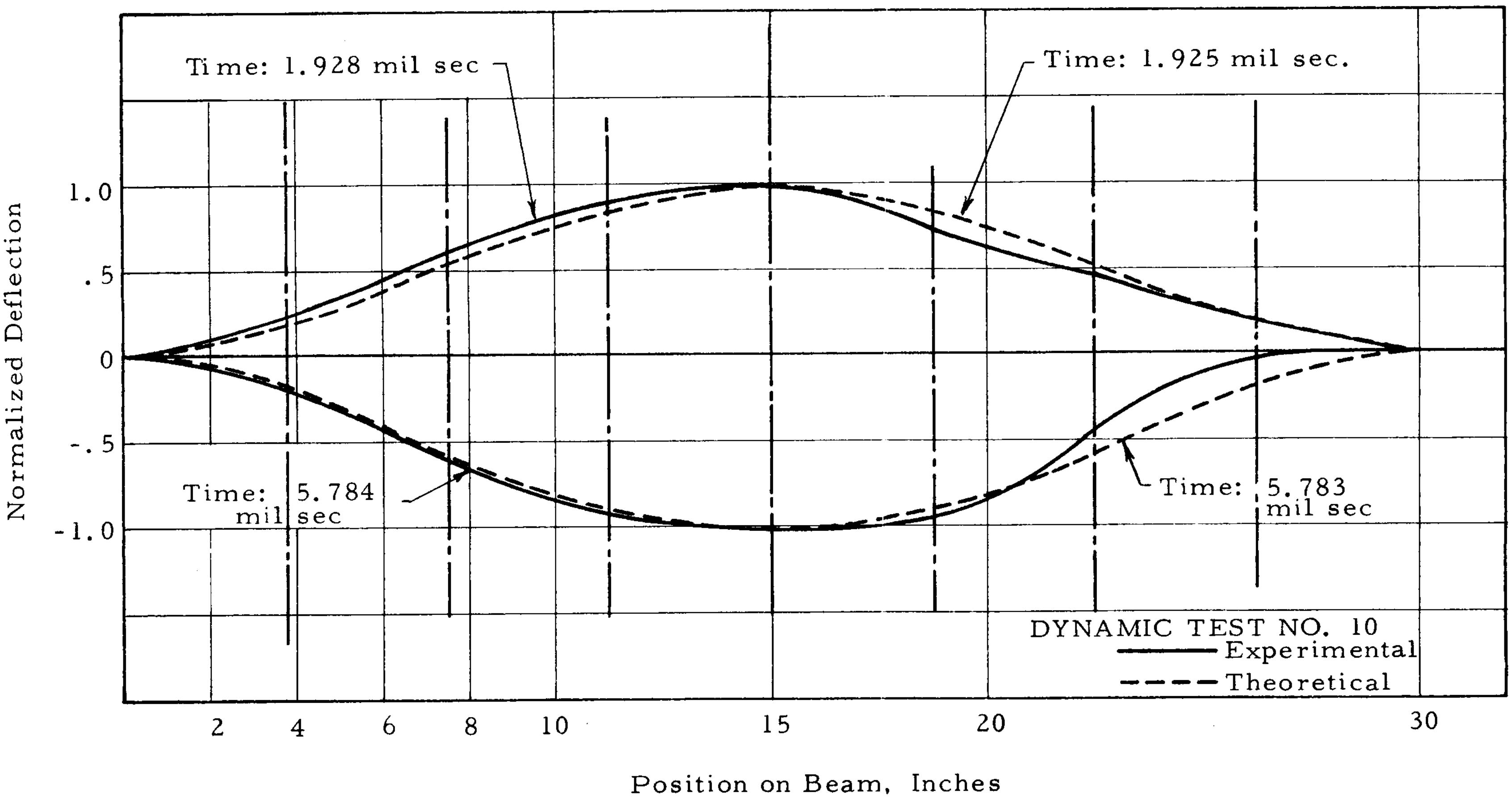

FIGURE 15. SHAPE OF VIBRATING BEAM-CLAMPED BEAM

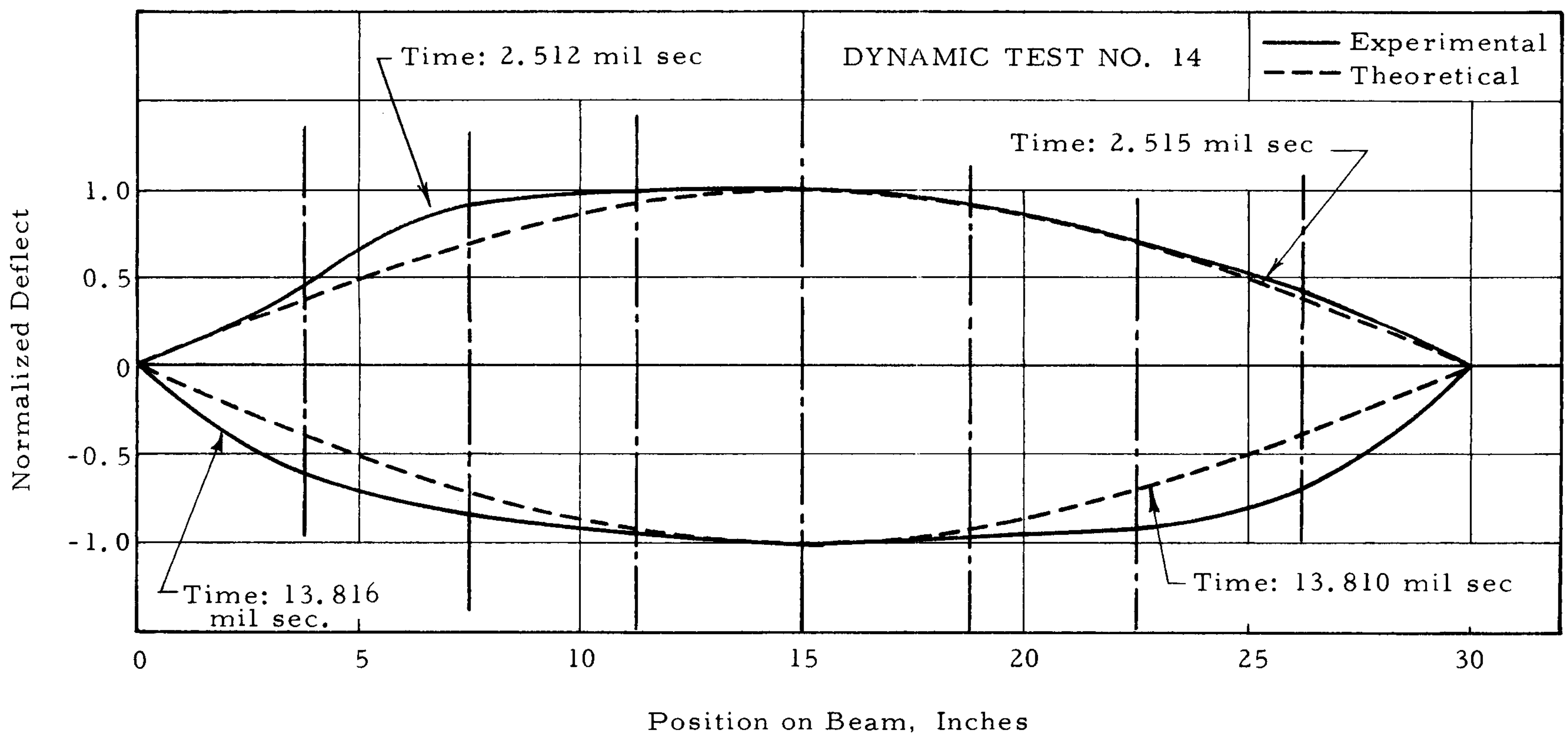

FIGURE 16. SHAPE OF VIBRATING BEAM-SIMPLE BEAM

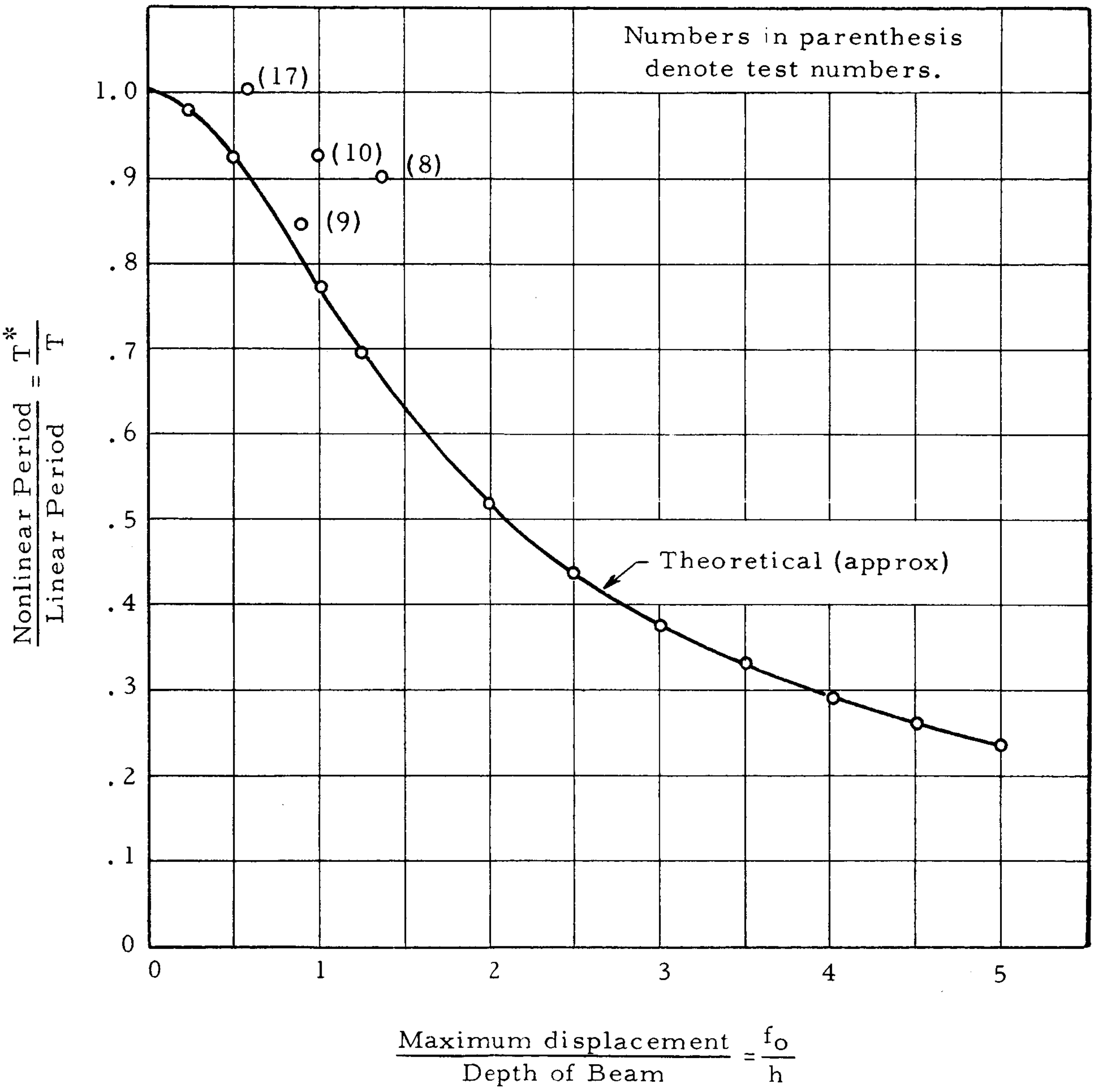

FIGURE 17. PLOT OF PERIOD RATIO AGAINST DISPLACEMENT RATIO-CLAMPED BEAM

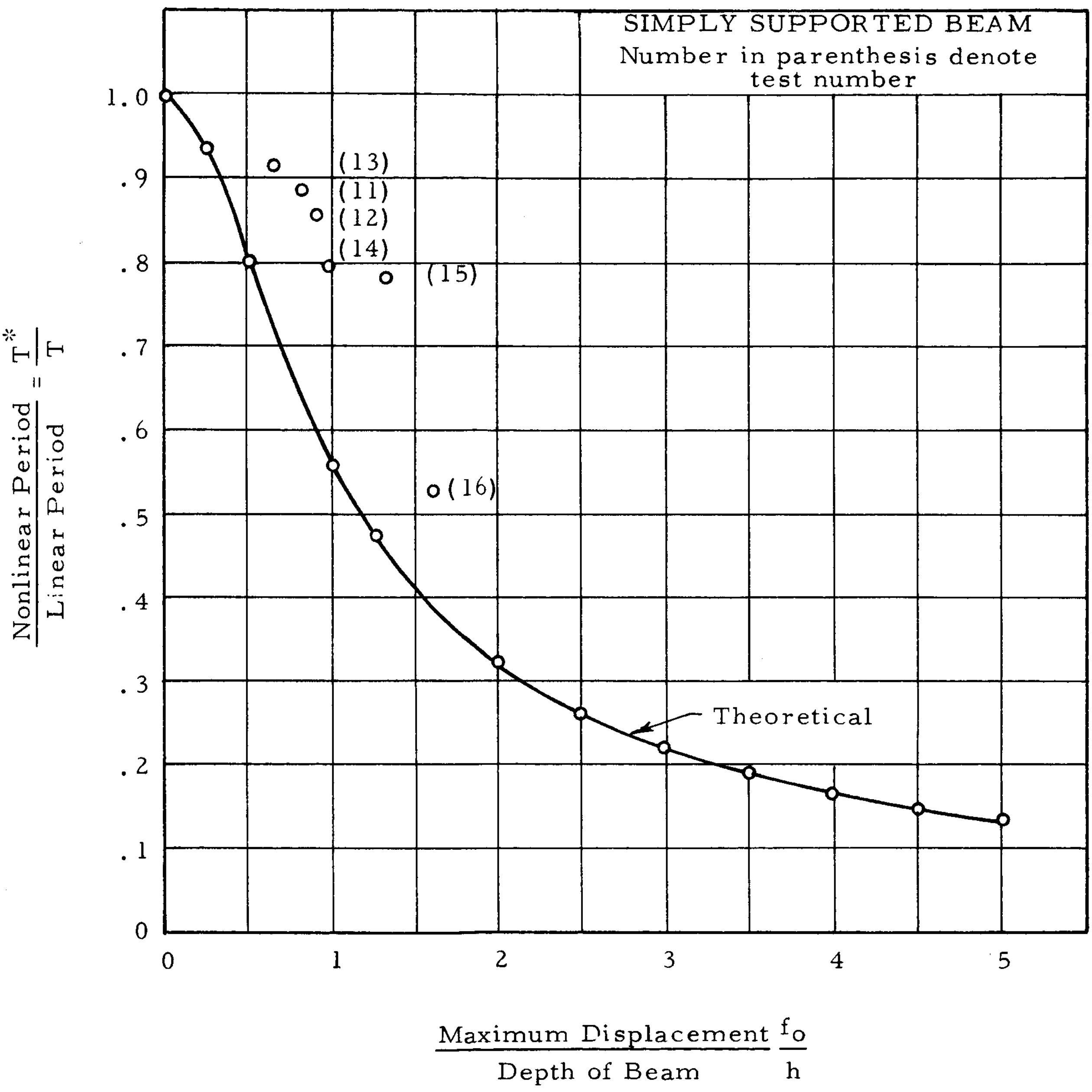

FIGURE 18. PLOT OF PERIOD RATIO AGAINST DISPLACEMENT RATIO-SIMPLE BEAM

Finally, Figures 17 and 18 show a plot of the ratio $\frac{T^*}{T} = \frac{\text{Nonlinear Period}}{\text{Linear Period}}$ against the amplitude ratio $\frac{f_o}{h} = \frac{\text{Maximum Displacement}}{\text{Depth of Beam}}$. It is seen that while the trend of the curve is reasonably well duplicated, especially in the simply supported case, the agreement between theory and experiment is not satisfactory.

VI. CONCLUSIONS

The experimental program has involved all kinds of difficulties that were not foreseen,leading to rather unsatisfactory results. Despite this, the present investigator is confident that all the problems encountered in experimental technique can be overcome by suitable modifications.

It appears essential for the further development of the Moiré method that one should be equipped with a fast as well as fine grained film. Generally, the method can be applied with sufficient accuracy only to structures with fairly large reflective surfaces such as plates.

The load application method needs to be modified and improved. It is obvious that a sinusoidal excitation will be of little value in locating the natural frequencies because of the possibility of subharmonic resonance, among other reasons. Possibly, an elliptic sine type excitation, if one can be designed and applied, may lead to fruitful results.

The program makes it abundantly clear that an experimental investigation of nonlinear vibration characteristics is beset with many difficulties and several methods require patient and possibly prolonged investigation.

The theoretical studies established the stability of "time dependent normal modes" for systems where space and time variables are not separable. It is believed that future progress in this area must lie in the development and refinement of numerical integration procedures for such systems.

LIST OF REFERENCES

1. Thein Wah, "The Normal Modes of Vibration of Certain Nonlinear Continuous Systems," Journal of Applied Mechanics, ASME, March 1964, p 139.

2. Thein Wah, "Large Amplitude Flexural Vibration of Rectangular Plates," International Journal of Mechanical Sciences, December 1963.

3. Thein Wah, "Vibration of Circular Plates at Large Amplitudes," Proc. ASCE, EMD Journal, October 1963.

4. R.M. Rosenberg, "The Normal Modes of Nonlinear N-Degree of Freedom Systems," Journal of Applied Mechanics, Vol 29, Trans. ASME, Vol 84, Series E, March 1962, pp 7-14.

5. W.A. Bradley, "Laterally Loaded Thin Flat Plates," ASCE Transactions, 1961, Vol 126, pp 875-915.

6. S.O. Durrant, "Moiré Patterns and the Analysis of Laterally Loaded Thin Plates," Presented at the Great Salt Lake Section Meeting of the Society for Experimental Stress Analysis, September 17, 1963.